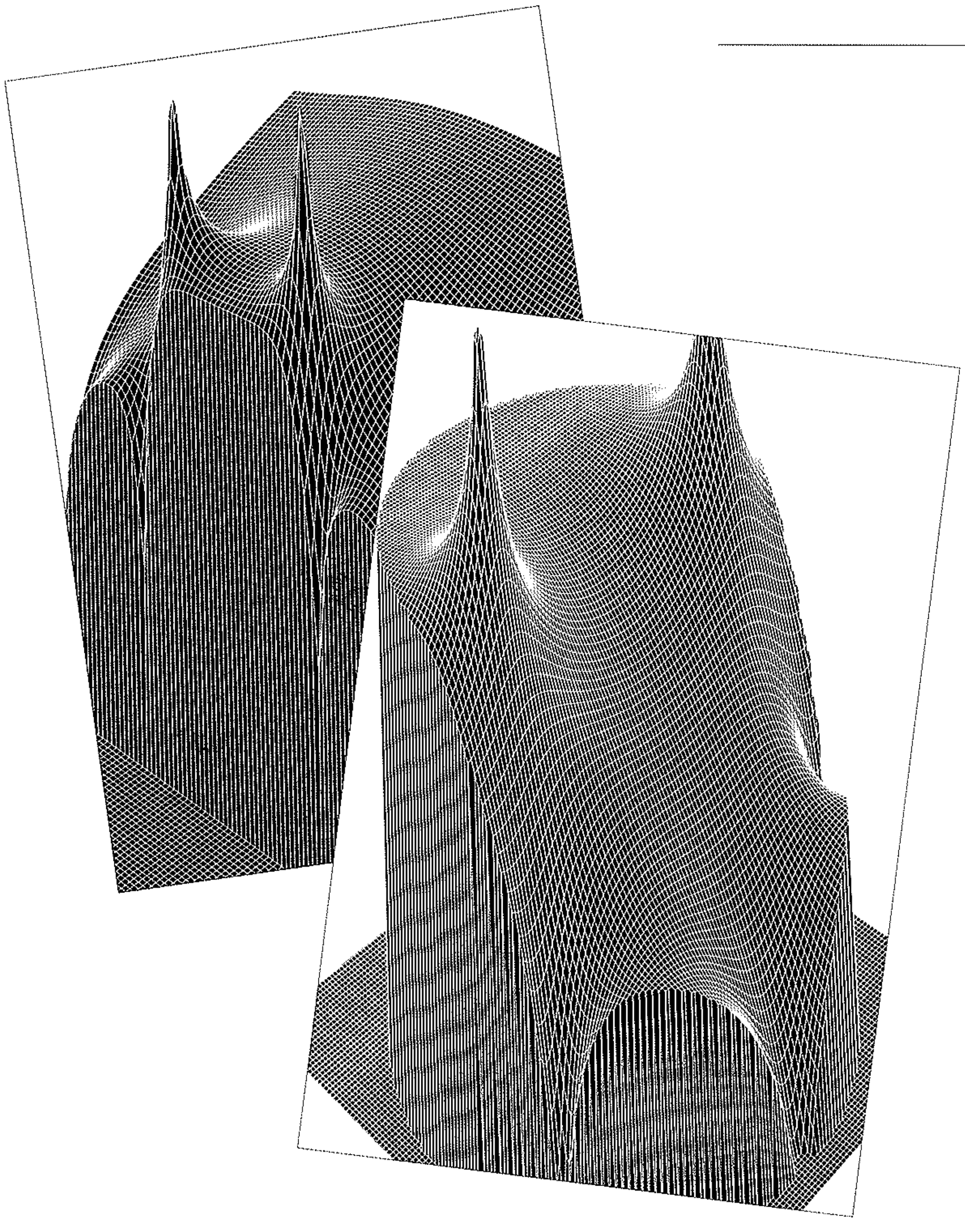
HEWLETT PACKARD

Fundamentals of the z-Domain and Mixed Analog/Digital Measurements

Application Note 243-4



Preface

There are numerous applications involving mixed analog and digital signals in the same system. In order to make measurements on mixed systems of this sort, it is helpful to use the z-transform for the digital part in conjunction with the Laplace transform (s-domain) for the analog part. In this note, the z-transform is defined and various transformations between the s and z domains are discussed. The Appendix is devoted to a discussion of matching the impulse responses of multiple poles in both the s and z domains.

The key characteristics of mixed domain measurements are also discussed in this note. For example, multiple images occur in the spectrum of a sampled signal. To measure the higher order images of the digital transfer function with a dynamic signal analyzer, the analog sampling rate is generally some integer multiple of the digital rate. To accurately measure the frequency response of a mixed system, these two sampling rates must be carefully locked together in both frequency and phase. There is also the need to handle time delays, both in the signal path and in the sampling pulse path.

Contents

Introduction

There are many applications in which signals are represented in both analog and digital form at different nodes in a system. For instance, control systems in which some part of the control loop is implemented in digital form, such as the loop compensation, are becoming more common. Figure 1 shows a simple block diagram of a mixed domain system which contains analog filters, data converters, and a digital filter. Measurements in mixed systems of this sort are somewhat more complicated than those for strictly analog systems since at least one of the time waveforms is only available in sampled form.

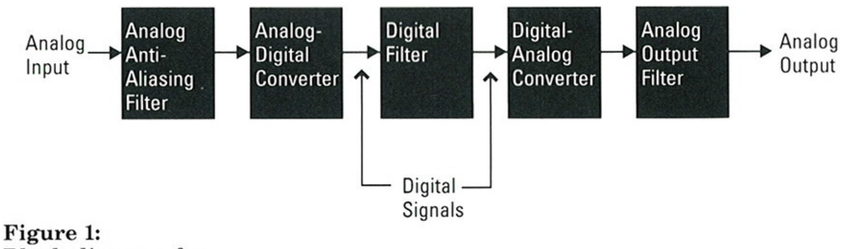


Figure 1: Block diagram for a mixed analog/digital system.

The frequency response function of a sampled data system is periodic along the frequency axis, with images spaced at multiples of the sampling rate (see figure 8 for an example showing four images). This implies that poles and zeros in the original s-domain are also replicated along the frequency axis, resulting in an infinity of new poles and zeros at multiples of the sampling frequency. The analog part of a mixed domain system is generally designed to suppress frequencies corresponding to these higher order images in the digital domain. This is the purpose of the analog anti-aliasing and output filters in figure 1.

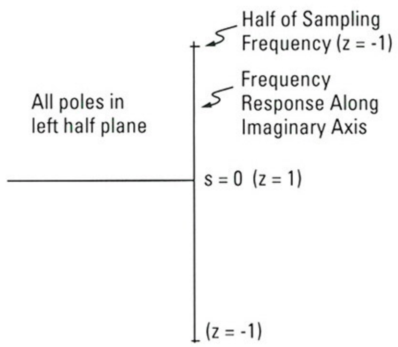


Figure 2: The s-domain, showing the frequency (imaginary) axis.

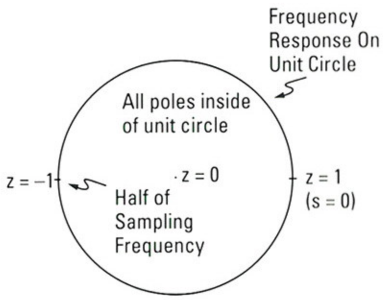


Figure 3: The z-domain, showing the unit circle that corresponds to the frequency axis in the s-plane.

The z-transform is used to characterize the transfer function of a sampled data system. This transform will be derived later, but it is simply a technique for representing a periodic frequency response function around a circle instead of along the linear frequency axis in the s-plane. Each frequency image is mapped onto one cycle around the unit circle. The values of the z-transform around the unit circle correspond to the measured frequency response function (at least below half of the sampling frequency), just as the values of the Laplace transform along the imaginary axis correspond to the measured frequency response function. All poles and zeros in the left half of the s-plane map into the interior of the unit circle in the z-plane, and the entire right half of the s-plane maps into the exterior of the unit circle.

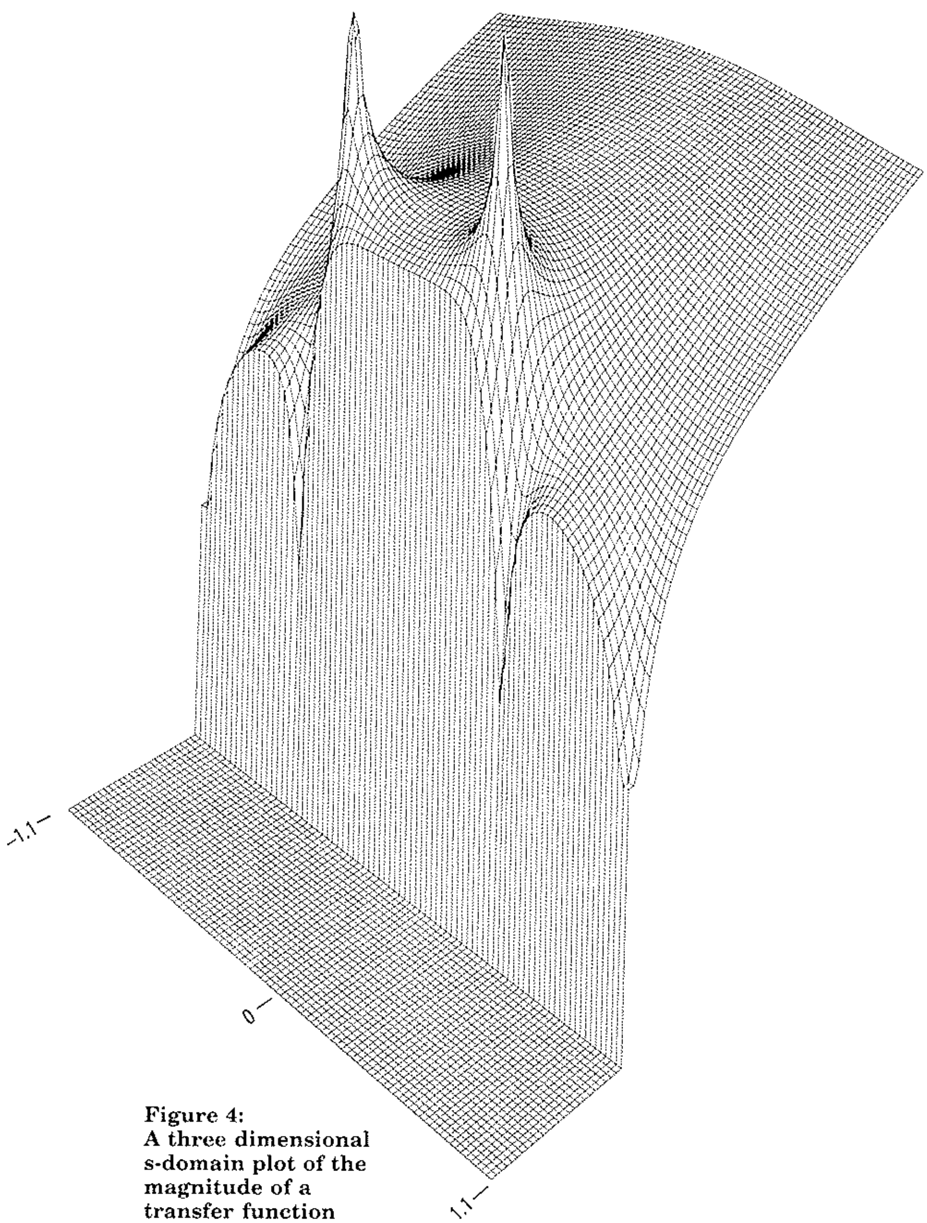


Figure 4:
A three dimensional s-domain plot of the magnitude of a transfer function having two poles and four zeros, with the right half of the plane cut away to show the frequency response function along the imaginary axis.

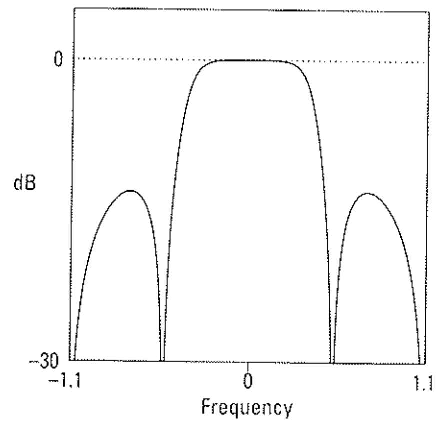


Figure 5:
A plot of the frequency response function illustrated in figure 4.

As a brief review and comparison, figure 2 shows the conventional s-domain, and figure 3 shows the z-domain with the unit circle drawn. Figure 4 shows a three-dimensional view of an analog filter in the s-domain, with the right half of the plane removed, showing the frequency response function along the imaginary axis. Figure 5 is a plot of this frequency response function. This filter comprises a pair of poles in the left half plane, along with 4 zeros along the frequency axis.

Figure 6 shows a three-dimensional z-domain view of a digital filter with characteristics similar to those of the analog filter. Here, the outside of the unit circle has been removed to show the frequency response function around this circle. Figure 7 shows a plot of this frequency response function. This filter comprises a pair of poles inside of the unit circle forming the filter pass-band, and three zeros on the unit circle forming the stop-band.

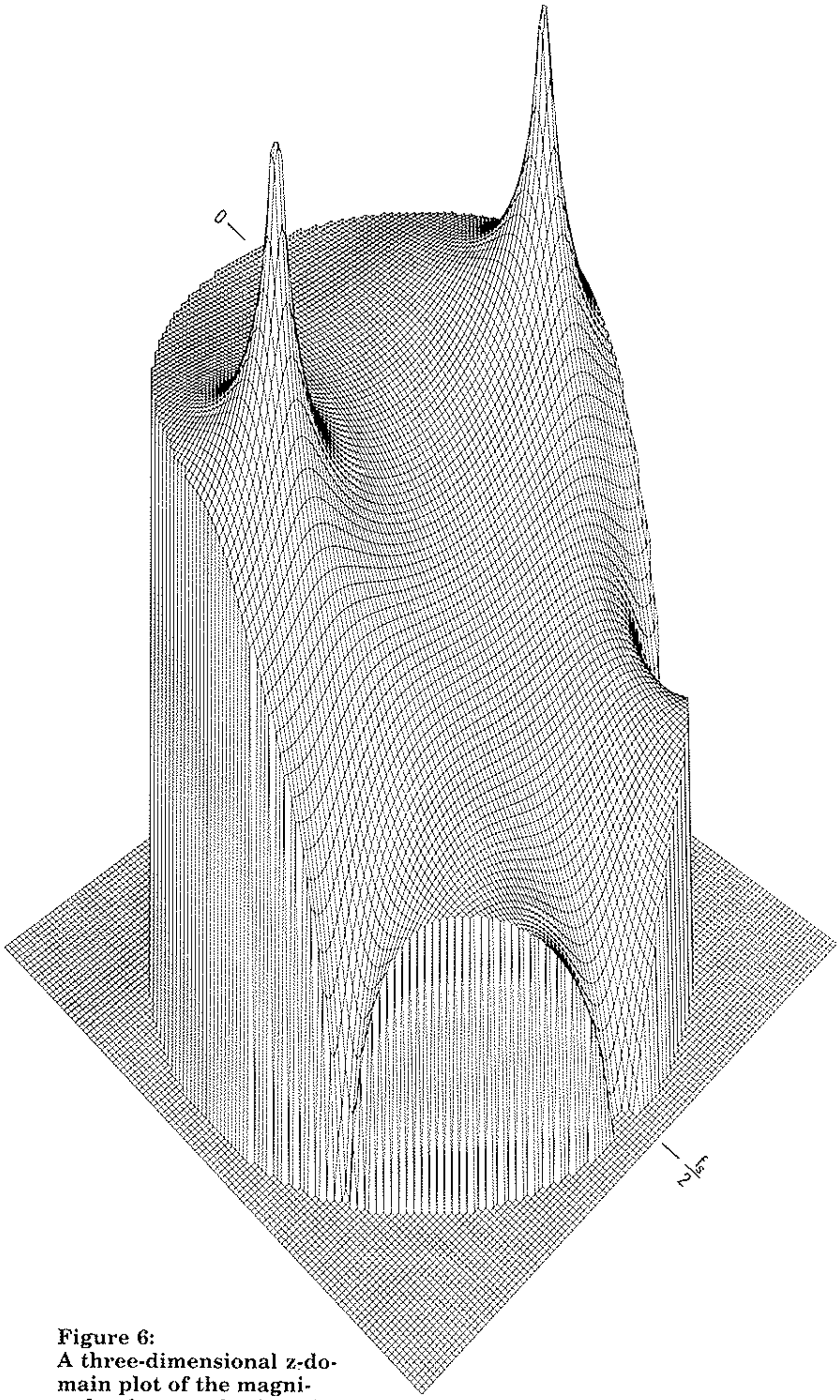


Figure 6:
A three-dimensional z-domain plot of the magnitude of a transfer function having two poles and three zeros, with the exterior of the unit circle cut away to show the frequency response characteristic.

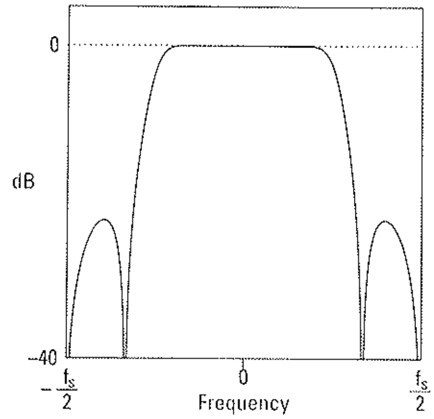


Figure 7:
A plot of the frequency response function illustrated in figure 6.

The z-domain is particularly useful for representing the behavior of digital filters, which involve combinations of adders, multipliers, and sample delay registers, since $1/z$ represents one sample of time delay. The equation describing the performance of a digital filter comprises the ratio of two polynomials in z, just as the performance of an analog filter comprises the ratio of two polynomials in s.

In the next section, the z-transform is derived from the Laplace transform and various techniques for converting from one domain to the other are discussed. In the final section, the differences between the two domains will be discussed, along with some of the problems that are encountered in mixed domain measurements.

Chapter 1: Derivation of the z-Transform from the Laplace Transform

The Laplace transform H(s) of some system whose impulse response is h(t) is given by

$$H(s) = \int_0^\infty h(t) e^{-st} dt \qquad (1)$$

where s is the Laplace variable. The impulse response is assumed to be zero for negative time values.

In a sampled data system with a sampling rate of f_s, the sample interval in the time domain is $\Delta t = 1/f_s$. A sampled version of h(t) can be obtained by multiplication by the "Shah" function (see reference [1]) defined by

$$III(t/\Delta t) = \Delta t \sum_{k=-\infty}^{\infty} \delta(t - k\Delta t) \qquad (2)$$

where $\delta(t-k\Delta t)$ is the unit impulse or delta function centered at $t = k\Delta t$. The area under this delta function is unity. If this sampled version of h(t) is inserted into (1), and the orders of integration and summation are interchanged, the resulting s-domain transfer function for this sampled system becomes

$$H(s) = \Delta t \sum_{k=0}^{\infty} h(k\Delta t) e^{-ks\Delta t} \qquad (3)$$

Make the substitution

$$z = e^{s\Delta t} \qquad (4)$$

Then the z-transform of the system impulse response is

$$H_z(z) = \frac{H(s)}{\Delta t} = \sum_{k=0}^{\infty} h(k\Delta t) z^{-k} \qquad (5)$$

The quantity z^{-k} is the Laplace transform of a delta function delayed by $k\Delta t$ in the time domain. The coefficient on the kth power of 1/z is simply the kth sample of the impulse response. Note that the sampling interval Δt has been removed as an amplitude multiplying factor from the definition of the z-transform in (5). This factor must be restored to evaluate the frequency response along the unit circle.

The periodic nature of the transform of sampled time data along the frequency axis can be seen from (3), where an exponential in continuous time has been replaced by an exponential involving multiples of the sampling interval Δt. Whenever s is replaced by $s+i2\pi n/\Delta t$, for any integer n, the value of the transform is unchanged. Figure 8 shows the frequency response for a simple pole at $s = -0.1$ (solid curve), and the four images obtained by evaluating the z-transform of the impulse response around the unit circle (dashed curve).

Equation (4) completely defines the z-domain in terms of the s-domain, and it is apparent that there is no new information about the transfer function contained in the z-domain representation. In fact, the z-domain form actually contains less information than the original s-domain form, to the extent that the original frequency response bandwidth exceeds half of the sampling rate. Any higher frequency components have been replaced by periodic replication of the lowest order image or, from another perspective, continuous time data has been replaced by sampled data. The loss of information is also apparent from equation (4), where a value for z is always uniquely determined for any given value of s, but the converse is not true. Thus, the merit of the z-domain is that it only shows the available information about the transfer function, whereas the s-domain may show redundant information.

Unfortunately, if equation (4) is used to obtain the z-transform directly from the Laplace transform, a rational fraction in s (comprising poles and zeros in the s-domain) becomes a transcendental function in z. For example, a simple pole in the s-domain can be written in the z-domain as

$$\frac{1}{s+a} = \frac{\Delta t}{ln(z) + a\Delta t} \qquad (6)$$

Any hardware implementation of a z-domain digital filter comprises various combinations of adders, multipliers, and sample delays represented by integer powers of 1/z. Thus, the z-domain form of the system transfer function must comprise finite order polynomials in z, and hence can be represented either as a rational fraction or a partial fraction in z. The transcendental form shown in (6) cannot be easily implemented physically. This argument implies that any practical transformation between the s and z domains must be only approximate. This raises the question as to the amount of error introduced by the approximation. Some of the more common transformations between these two domains will be discussed next, and some examples of the associated errors will be given.

Chapter 2: Transformations Between s and z Domains

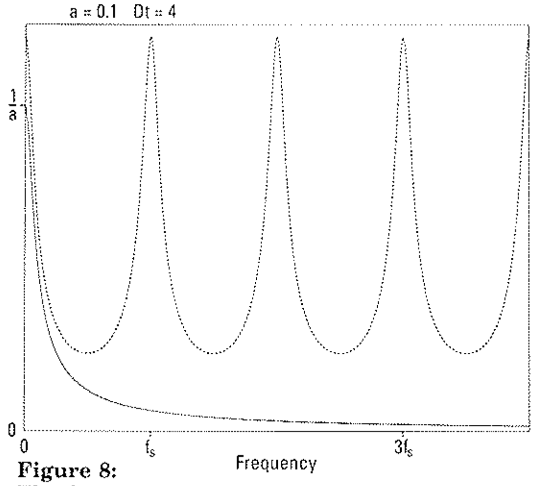


Figure 8:
The frequency response function of a simple pole in the s-domain (solid line), compared with the frequency response of the impulse invariant form of the z-transform (dashed line). Note the four images introduced by the sampling operation. Also note the error in peak amplitude.

There are two generic types of transforms between the s and z domains, with numerous variations on each method. The first type involves matching time waveforms, usually either the system impulse response or the step response. The second type involves rational fraction approximations to equation (4), such as given by the bilinear transformation.

None of the above methods are exact, and the choice between them depends upon the application at hand. The common link between analog and digital parts of a mixed system is either the frequency response function or the impulse response in the time domain. Thus, the significance of any errors introduced by approximations between the s and z domains will ultimately be viewed along either the frequency or the time axis. Because of aliasing in sampled systems, it is often not possible to match both the frequency response function and the impulse response simultaneously. In general, either the impulse invariant or the step invariant methods are best when the time response is of interest. The bilinear transform is used for frequency response matching, but is only accurate for very low frequencies, relative to the sampling frequency f_s.

2.1: Impulse Invariant Transformation

The impulse response can be matched by decomposing the transfer function in either the s or the z domain into partial fractions, then matching the impulse response of each term. This is easy to do if the multiplicities of all poles are unity, but becomes more complicated for multiple poles. The multiple pole case is discussed in the Appendix. A simple pole is represented in the s-domain by

$$H(s) = \frac{1}{s+a} \tag{7}$$

and the corresponding impulse response is

$$h(t) = e^{-at},\ for\ t > 0 \tag{8}$$

From (5), the z-transform can be written as

$$H_z(z) = \sum_{k=0}^{\infty} e^{-ak\Delta t} z^{-k} \tag{9}$$

$$= \frac{z}{z - e^{-a\Delta t}} \tag{10}$$

Thus, each partial fraction term in the s-domain with a pole at s = –a yields a partial fraction term in the z-domain, with a zero at the origin and a pole at z = exp(–aΔt). The sampled values of the impulse response become the coefficients on an infinite series in 1/z, as shown in equation (9), which can be written in closed form (10).

The frequency response corresponding to the z-domain transfer function is obtained by multiplying H_z by Δt, for z=exp(i2πfΔt). If Δt is sufficiently small, then this becomes

$$H(i2\pi f) = \frac{1}{a + i2\pi f} \tag{11}$$

which is the same as obtained from the s-domain via (7). However, when Δt is not sufficiently small, the z-domain frequency response is different from the s-domain response. This is a direct result of the aliasing that occurs in the frequency domain when images of the frequency response function are replicated at multiples of the sampling frequency.

2.2: Step Invariant Transformation

In a similar manner, it is possible to match the response to a unit step. The s-domain transfer function is multiplied by 1/s, and the result is expressed in partial fraction form. Then, each term is converted to the z-domain, as indicated above, and multiplied by (z-1)/z to remove the input step.

If this technique is used to match the step response for a simple pole, as given by (7), the result is

$$H_z(z) = \frac{A}{z - e^{-a\Delta t}} \tag{12}$$

where

$$A = \frac{1 - e^{-a\Delta t}}{a\Delta t} \tag{13}$$

Compared to (10), this transfer function has only a pole at z = exp(–aΔt), and no finite zeros.

None of these techniques that match responses in the time domain consider the effects of aliasing caused by undersampling. Thus, even though the time response is matched at the sample values, any waveform details that may occur between samples, such as fast level transitions or narrow pulses, are lost. This implies that the higher frequencies in the frequency response function may be in error to some degree. This is a direct consequence of the potential overlap between the replicated frequency images that result from time domain sampling.

Only partial fraction terms that involve poles or a constant can be precisely converted from one domain to another. Thus, any higher order polynomial components that result from the partial fraction expansion cannot be converted. This means that the order of the numerator of the rational fraction form must be no greater than the order of the denominator for either an impulse invariant or a step invariant conversion to exist. An exception to this rule can be made for any powers of z that can be removed from the rational fraction before conversion, since these powers of z can be represented as time advances. This is also true for powers of 1/z which can be represented as time delays.

2.3: Bilinear Transformation

The second type of transformation between the s and z domains is used when frequency response function matching is needed at low frequencies, and involves some sort of rational fraction approximation to equation (4). The most common approximation is called the bilinear transform, which is obtained from the quotient of two first order polynomials in s. Equation (4) can be written as

$$z = e^{s\Delta t} = \frac{e^{s\Delta t/2}}{e^{-s\Delta t/2}} \tag{14}$$

If only the first two terms in the Taylor's series expansion of the numerator and the denominator are retained, then z can be approximated by

$$z \cong \frac{1 + s\Delta t/2}{1 - s\Delta t/2} \tag{15}$$

This can be inverted to obtain

$$s \cong 2/\Delta t \frac{z-1}{z+1} \tag{16}$$

Equation (16) is called the bilinear transform (the quotient of two linear expressions), and (15) is sometimes called the inverse bilinear transform. This form has the advantage of limiting the orders of the z-domain polynomials to the maximum order of the s-domain polynomials. Obviously, there are many other possible polynomial approximations to (4), but this is the one most often used in practice.

The bilinear form also has the property of mapping the entire s-domain frequency axis onto the unit circle in the z-domain, in contrast to the exact definition of z, in which only frequencies up to half of the sampling rate are mapped onto the unit circle. Unfortunately, this mapping results in a considerable amount of frequency "warping", especially for frequencies near the point $z = -1$. This warping is described by

$$f' = \frac{\tan(\pi f \Delta t)}{\pi \Delta t} \tag{17}$$

where f' is the frequency after the bilinear transform has been imposed, and f is the frequency around the unit circle in the z-domain at which f' is mapped. Note that f' becomes infinite when $f = 1/(2\Delta t)$ = half of the sampling frequency.

For purposes of comparison, the expression for s in (16) can be substituted into (7) to obtain the bilinear form for a simple s-domain pole. The result is

$$H_z(z) = \frac{H(s)}{\Delta t} = B\frac{z+1}{z-b} \tag{18}$$

where

$$B = \frac{1}{2 + a\Delta t} \tag{19}$$

$$b = \frac{2 - a\Delta t}{2 + a\Delta t} \tag{20}$$

Thus, a pole is placed at $z = b$ and a zero is placed at $z = -1$. If the sampling interval Δt is sufficiently small, z can be replaced by $1+i2\pi f\Delta t$ and b is approximately $1-a\Delta t$, so $H(i2\pi f)$ becomes $1/(a+i2\pi f)$, as expected. However, a comparison of (18) with (10) shows that these equations are not equivalent and, therefore, the frequency response and the impulse response will be different.

These transformation techniques that involve approximations to (4) tend to include the effects of aliasing to some extent, but the resulting responses in the time domain may not be very accurate. These approximations are only good for small values of s, for which z is near unity.

2.4: Representation of Time Delays

Any time delay in the s-domain representation of a system must be carried as a separate parameter since there is no finite rational fraction representation of this delay. However, in the z-domain, integer multiples of the sampling interval Δt are represented as powers of $1/z$, which are simply poles at the origin in the z-plane. Thus, these discrete time delay values can be represented as part of a z polynomial. Unfortunately, this technique does not work for time delays that are fractions of the sampling interval, so it is still necessary to carry a time delay parameter separately. One possible convention is to always represent the integer time delay multiples of Δt as z-domain poles at the origin, and to represent only the fractional part of the delay as a separate parameter. However, this is an arbitrary choice, and other conventions for representing delay are equally valid.

In any case, when a z-domain transfer function is converted into an s-domain representation, the resulting time delay is the sum of the part represented by a multiple pole at the origin of the z-domain, and the part represented as a separate delay parameter.

2.5: Comparison of Different Transformation Techniques

The results of each transformation technique can be compared by viewing the amplitude and phase response in the frequency domain, and/or the impulse response in the time domain. Figure 9 shows the amplitude frequency response of a simple pole in the s-domain for $a = 0.1$ (solid curve), along with curves evaluated from three z-domain approximations (dashed curves). The upper dashed curve is for the impulse invariant transformation, and the middle dashed curve is for the step invariant case. The lower dashed curve is for the bilinear transform. Notice the zero at $z = -1$ for the bilinear case. Also note that the dc value of the response is not correct for the impulse invariant case, although when this is normalized away, this curve coincides with the step invariant curve.

Figure 10 shows the phase response for the same four cases illustrated in figure 9. The solid line represents the phase for the s-domain representation of a simple pole, while the dashed lines represent the phase for three different z-domain approximations. It is only necessary to consider the phase angle for positive frequencies below half of the sampling rate (left half of the figure) since the negative frequency interval will be symmetric. The best phase match to the solid line is obtained by means of the bilinear transform (middle dashed line). This is expected since the bilinear transform incorporates, to some extent, the effects of aliasing. The upper dashed line is for the impulse invariant case, and the lower dashed line is for the step invariant case. The step invariant case incorporates an extra phase slope that corresponds to one sample of delay. If this delay is removed, this case is identical to the impulse invariant case.

Figure 11 shows the impulse responses in the time domain for these same four cases. The solid line is the continuous time impulse response for a simple pole, and the labels $*$, o, and x show the sampled versions of this impulse response for the impulse invariant method ($*$), the step invariant method (o), and the bilinear transformation (x). The impulse invariant method gives exact sample values. If the step invariant results were re-scaled in amplitude, they would also be correct except for one sample of delay. The bilinear transform results need to be scaled in amplitude, and the decay time constant is also slightly in error (too small by 1.348%, for this case).

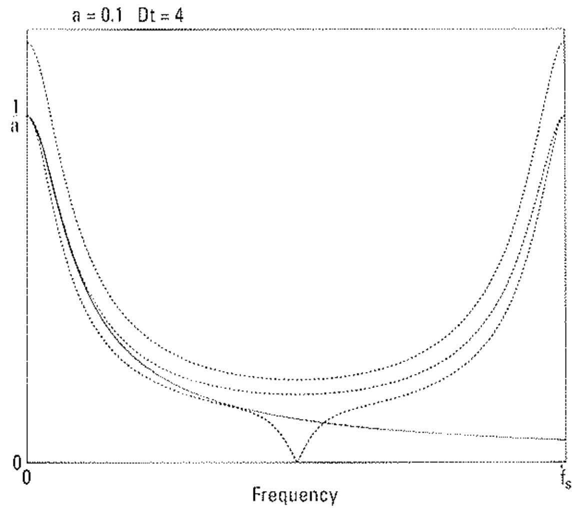


Figure 9:
The frequency response function of a simple pole in the s-domain (solid line), compared with the frequency responses for three different z-domain representations. The upper dashed line is for the impulse invariant case, and the middle dashed line is for the step invariant case. The bilinear transform case is shown by the lower dashed line. When the upper line is scaled to be correct at dc, it matches the middle line at other points, as well.

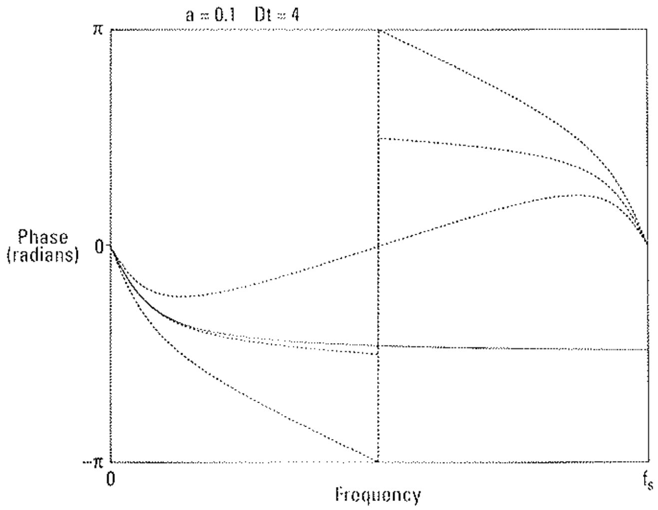


Figure 10:
The phase response of a simple pole in the s-domain (solid line), compared with the phase responses of three different z-domain representations. Only the left half of the plot is useful for this comparison. The upper dashed line is for the impulse invariant case, and the lower dashed line is for the step invariant case. The middle line is for the bilinear transformation.

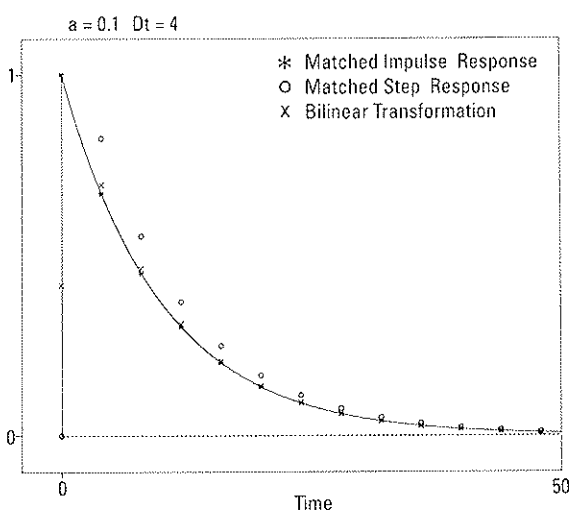


Figure 11:
The impulse response of a simple pole in continuous time (solid line), and for sampled times corresponding to the impulse invariant method (* symbol), the step invariant method (o symbol), and the bilinear transform technique (x symbol).

Chapter 3: Characteristics of Mixed Domain Measurements

When continuous time and sampled time systems are connected together as shown in figure 1, there arises the need to make frequency response measurements across the interface between the two domains. When making measurements in a mixed analog/digital system, the key characteristics to be aware of are:

- The occurrence of multiple spectral images of sampled signals
- The need to synchronize analog and digital sampling rates
- The possible presence of two types of time delays

In mixed domain systems, multiple images occur in the spectra of sampled signals. It is generally necessary to filter the input signals to reduce aliasing, and to filter the output signals to attenuate the spectral images. The bandwidth of the measurements of analog signals must extend beyond the frequency of the highest image of concern.

There is also a need to synchronize the analog and digital sampling rates to avoid errors due to leakage. In addition, these sampling signals must be phase locked so that transfer functions between digital and analog parts of a system can be measured accurately. If there is any relative jitter between these two sampling signals, then additional errors will be introduced.

Two types of time delay appear in a mixed mode measurement, in contrast to only one type of delay in an analog measurement. In either case there can be a delay in the system impulse response, but there can be an additional delay in the sampling pulses for the digital part of a system. These two delays affect the results in different ways.

3.1: Images and Analog Filtering

The distinguishing feature of mixed domain measurements is the occurrence of multiple images in the spectrum of a sampled signal. Generally, a designer is interested in the effect that an analog filter circuit has upon the multiple images introduced by the digital portion of a system. An ADC is an example in which aliasing is introduced into the primary spectral image if any input signal components occur at frequencies above half of the sampling rate. Attenuating such unwanted signals is the purpose of the low-pass anti-aliasing filter in figure 1. To observe higher frequency signal components, the bandwidth of a measurement on the analog input to an ADC must extend beyond the frequency encompassed by the highest image of concern in the sampled signal.

In a similar manner, the analog filter on the output of the DAC in figure 1 is designed to attenuate the higher order images coming out of the mixed system. This filter also serves to convert the discrete samples of the DAC output into a continuous analog signal. To show all of the attenuated images of interest, it is necessary to make analog measurements at frequencies higher than the digital sampling rate.

This latter filter must be designed to attenuate all of the frequency domain images of the spectrum except the one of interest. One common filter type is that obtained by means of a zero order hold circuit. The impulse response of this filter is a rectangle having a unit area and a width equal to the sample interval Δt. This gives a filter shape of $\sin(\pi f \Delta t)/(\pi f \Delta t)$, which has nulls at the center of each image except the one centered at the origin. This filter shape is shown as a dashed line in figure 12 and its effect upon the frequency images (of figure 8) is shown as a solid line. In addition, there will be a linear phase shift versus frequency corresponding to a delay of $\Delta t/2$.

Other filters must generally be added to further reduce the sizes of the unwanted images. It is apparent from figure 12 that these reduced images can still be relatively large. It is possible to use higher order hold circuits, corresponding to triangular or parabolic impulse responses, but it is usually easier to design one of the standard analog low-pass filters such as either the Chebyshev or elliptic types.

Often, mixed mode systems are designed so that the digital filter part complements the analog part to obtain better overall characteristics than could be obtained with either technique separately. For example, the digital filter might be designed with a narrow pass-band relative to the sampling frequency, or the sampling frequency might be multiplied, so that the subsequent analog filter can have a wider transition band between the desired image and the remaining rejected images. This oversampling allows use of a simple analog filter design, having well controlled phase and amplitude characteristics.

3.2: Synchronization of Sampling Pulses

The second key characteristic of mixed domain measurements is the need for synchronized sampling pulses between the two domains. Not only should the sample rates be related by simple integers, but the relative phases between the two sampling signals must be known or measured so that mixed domain transfer functions can be determined.

Generally, the analog sampling rate will be some integer multiple of the digital rate. However, the digital rate is often determined by the device under test, so the analog sampling signal must be derived from the digital rate in some manner. To minimize leakage effects when working with periodic signals, the analog sampling rate must be a very accurate multiple of the digital rate. Thus, the digital rate must be known or measured very accurately, and the analog sampling rate must be very accurate and stable in frequency.

There are two types of errors that can occur when timing differences exist between the analog and digital sampling signals. The first type of error is due to a discrepancy in the average analog sampling rate (not exactly an integer multiple of the digital rate). In this situation, a periodic digital signal will not remain exactly periodic after being sampled at the analog rate. When using a control systems analyzer or dynamic signal analyzer to make measurements in this situation, the sampling delays can cause leakage errors in the frequency spectrum, especially if the user selects a rectangular or "uniform" window. For example, when making distortion measurements using a sinusoidal input, a uniform window is generally used and all harmonics are expected to be exactly periodic in the time window. Any leakage that occurs will directly affect the accuracy of measurements of the higher order harmonics.

The second type of error is due to jitter on the digital sampling signal. There are times when this same jitter should also occur on the analog sampling signal. For example, if the transfer function of a DAC is being measured, then any jitter on the digital samples should be exactly duplicated on the analog samples so that the measured transfer function is independent of this jitter. However, if a digital compensator is embedded in a control system and there is some amount of jitter on the internal digital clock, then the analog sampling rate should probably be uniform in time so that the effects of the digital jitter can be observed.

Jitter on the digital sampling signal can also result in leakage errors, especially if a uniform time window is used in the measurement. For example, if the transfer function of a DAC with zero order hold is being measured using a uniform window and there is jitter on the digital clock, then leakage contributions from the higher order images will occur in the baseband frequency region, even if the analog sampling rate is much higher than the digital rate (negligible aliasing).

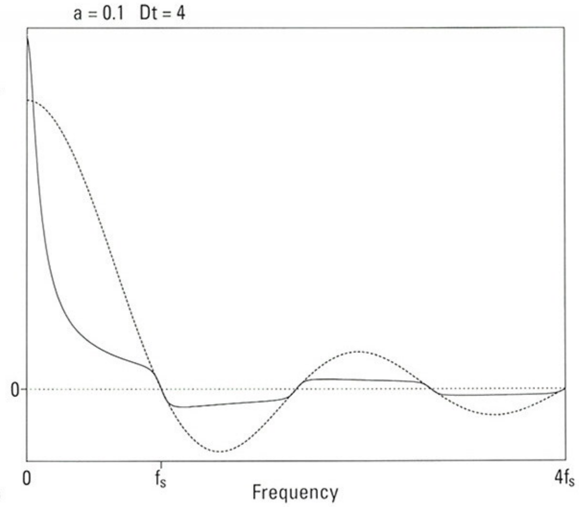


Figure 12: The equivalent filter of a zero order hold is shown as a dashed line, and the effect of this filter on the multiple frequency images of figure 8 is shown as a solid line. Note the nulls in this filter at the center of each image.

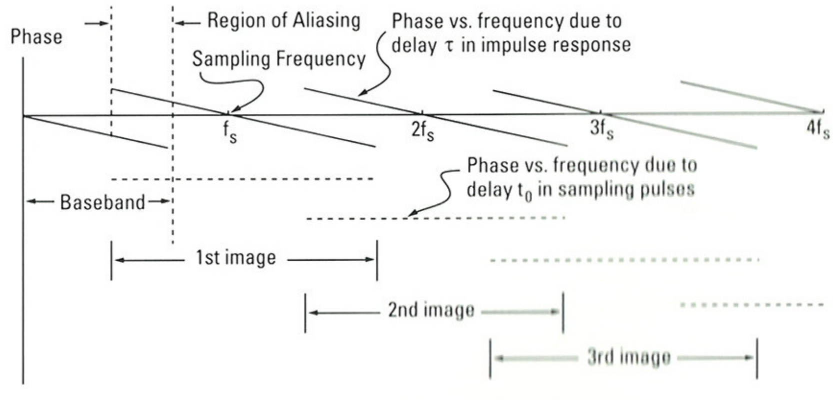


Figure 13: Phase versus frequency due to a time delay in the impulse response, and due to a time delay in the sampling pulses.

Chapter 4: Summary

3.3: Time Delays

The third major characteristic of mixed measurements that must be considered is the occurrence of time delays in the system. In the analog part of a system, a time delay results in a linear phase slope in the frequency response function and can be approximated by a rational fraction in the s-domain. In the z-domain, there are two types of time delays that must be treated separately. There can be time delays in the signal path, just as for analog systems, and there can be time delays in the sampling pulses, without any signal delay. In addition, both kinds of delay may occur simultaneously. Delays in the sampling pulses can occur if multi-phase clocks are used to perform several operations within one clock period, particularly if the output is clocked with a different phase than the input.

If there is a delay in the signal path, then the result is the same as for an ordinary analog delay. A linear phase slope is introduced into the frequency response function (see figure 13). A modified z-transform can be defined (see reference [2]) that matches the delayed impulse response, although the linear phase slope in the frequency response may not be correctly represented due to aliasing. Alternatively, a rational fraction in z can be used to approximate the phase slope, just as in the s-domain.

A delay in the sampling pulses only affects the phases of the higher order images of the frequency spectrum, and hence only affects the errors due to aliasing. If the original spectrum is band limited to half of the sampling frequency, then a delay in the sampling pulses has no effect upon the baseband spectral image (see figure 13). In the z-domain, the coefficients on the powers of 1/z are obtained from delayed samples of the impulse response, so the actual z-transform is modified by the sample delay. In addition, there is a factor of z^{-d}, where d is the sample time delay normalized by the sample interval Δt, which accounts for the phase differences among the frequency images.

When both the sampling pulses and the impulse response are delayed, the result is a combination of the effects discussed above for each separate delay. However, if both signals are delayed by the same amount, then the samples of the impulse response are the same as for no delay, and the resulting z-transform only differs by the z^{-d} factor defined above.

These time delay effects are best summarized by re-writing equation (5) for the z-transform, where h(t) has been replaced by $h(t-\tau)$, to represent a delay of τ in the impulse response, and t has been replaced by $t-t_0$ in the Shah function (equation (2)) to indicate a sampling pulse delay of t_0. The resulting z-transform can be expressed as

$$H_z(z) = \sum_{k=0}^{\infty} h(k\Delta t + t_0 - \tau) z^{-k-d}, \qquad (21)$$

$$d = \frac{t_0}{\Delta t}$$

Notice that these two types of delay enter into the equation in different ways, so their effects must be considered separately.

There are numerous applications involving mixed analog and digital signals in the same system. It is helpful to use the z-transform for the digital part, in conjunction with the Laplace transform (s-domain) for the analog part when making measurements on these mixed systems. The z-transform is defined and the impulse invariant, step invariant and bilinear transformations between the s and z domains are discussed and compared. The Appendix discusses the matching of the impulse responses of multiple poles in both the s and z domains.

Three key characteristics of mixed domain measurements are discussed: images and analog filtering; synchronized sampling; and time delay effects. Most mixed analog/digital systems contain analog filters on the input of ADCs to prevent aliasing and on the output of DACs to attenuate images. When using a control systems analyzer or dynamic signal analyzer to measure the higher order images in a mixed transfer function, the analog sampling rate should be some integer multiple of the digital rate. To make accurate frequency response measurements, these two sampling rates must be carefully locked together in both frequency and phase. There is also the need to handle time delays, both in the signal path and in the sampling pulse path.

Appendix: Multiple Pole Impulse Invariance

The transfer function of a multiple pole in the s-domain is

$$H(s) = \frac{1}{(s+a)^{k+1}} \quad \text{(A1)}$$

The pole is located at s = -a, and it has a multiplicity of k+1. The corresponding impulse response is given by

$$h(t) = \frac{t^k}{k!} e^{-at}, \quad for \ t \geq 0,$$
$$k = 0, 1, 2, \ldots \quad \text{(A2)}$$

The goal is to derive a z-domain representation that will exactly reproduce this impulse response at times sampled at Δt intervals. In particular, the sampled impulse response is given by

$$h(n\Delta t) = \frac{(n\Delta t)^k}{k!} e^{-an\Delta t},$$
$$for \ n = 0, 1, 2, \ldots \quad \text{(A3)}$$

This sampled impulse response can be generated from a z-domain formulation involving the sum of poles having all multiplicities from unity to k+1. The detailed derivation will not be given here, but the results for poles of multiplicity one through four will be shown. In general, whenever a pole of a given multiplicity occurs, all poles of lower order also occur. Thus, a matrix representation of this impulse invariant transformation is useful.

Define a normalized z-domain variable called x, as follows

$$x = \frac{e^{-a\Delta t}}{z} \quad \text{(A4)}$$

Define a four element vector S whose elements are the s-domain poles for each multiplicity. Define Z as 4-vector of z-domain poles of the form $1/(1-x)^{k+1}$ for each multiplicity. The elements for each of these vectors are listed in order of decreasing multiplicity. Then, the impulse invariant transformation between these two domains can be written in matrix form as

$$S \Leftrightarrow RZ \quad \text{(A5)}$$

where R is the 4x4 matrix

$$R = \begin{bmatrix} \Delta t^3 & 0 & 0 & 0 \\ 0 & \Delta t^2 & 0 & 0 \\ 0 & 0 & \Delta t & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 7/6 & -1/6 \\ 0 & 1 & -3/2 & 1/2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(A6)}$$

If a row vector A of s-domain coefficients on each element of S is defined as

$$A = [A_3 \quad A_2 \quad A_1 \quad A_0] \quad \text{(A7)}$$

then the final result can be written as

$$AS \Leftrightarrow ARZ \quad \text{(A8)}$$

In a similar manner, equation (A5) can be inverted to give

$$Z \Leftrightarrow R^{-1} S \quad \text{(A9)}$$

where the inverse of R is

$$R^{-1} = \begin{bmatrix} 1 & 2 & 11/6 & 1 \\ 0 & 1 & 3/2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta t^{-3} & 0 & 0 & 0 \\ 0 & \Delta t^{-2} & 0 & 0 \\ 0 & 0 & \Delta t^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(A10)}$$

If a row vector B of z-domain coefficients on each element of Z is defined as

$$B = [B_3 \quad B_2 \quad B_1 \quad B_0] \quad \text{(A11)}$$

then the result can be written as

$$BZ \Leftrightarrow BR^{-1} S \quad \text{(A12)}$$

These row vectors of coefficients are related by

$$B = AR \quad \text{(A13)}$$

If the multiplicity of the original pole is reduced by one, then the topmost row and the leftmost column of R (and of R^{-1}) are discarded to form a 3x3 R (and R^{-1}) matrix.

As for the unity multiplicity case, it is necessary to multiply the z-domain form of the multiple pole by Δt before attempting to calculate the frequency response function around the unit circle.

References

[1] Bracewell, R.N., The Fourier Transform and Its Applications, McGraw-Hill, 1986.

[2] Phillips, C.L., and Nagle, H.T., Digital Control System Analysis and Design, Prentice-Hall, 1984.

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